

# Stable 3-spheres in $\mathbb{C}^3$

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**Abstract:** By only using spectral theory of the Laplace operator on spheres, we prove that the unit 3-dimensional sphere of a 2-dimensional complex subspace of  $\mathbb{C}^3$  is an  $\Omega$ -stable submanifold with parallel mean curvature, when  $\Omega$  is the Kähler calibration of rank 4 of  $\mathbb{C}^3$ .

## 1. Introduction

In 2000, Frank Morgan introduced the notion of multi-volume for an  $m$ -dimensional submanifold  $M$  of a Euclidean space  $\mathbb{R}^{m+n}$ , as a volume enclosed by orthogonal projections onto axis  $(m+1)$ -planes. He characterized stationary submanifolds for the area functional with prescribed multi-volume as submanifolds with mean curvature vector  $H$  prescribed by a constant multi-vector  $\xi \in \wedge_{m+1} \mathbb{R}^{m+n}$ , namely  $H = \xi \lrcorner \vec{S}$ , where  $\vec{S}$  is the unit tangent plane of  $M$ , and proved the existence of a minimizer among rectifiable currents, as well as their regularity under general conditions of the boundary. In this setting, a question has arisen on conditions for  $\|H\|$  to be constant. In (Salavessa, 2010) we extended the variational characterization of hypersurfaces with constant mean curvature  $\|H\|$  to submanifolds with higher codimension, when the ambient space is any Riemannian manifold  $\bar{M}^{m+n}$ , as discovered by Barbosa, do Carmo and Eschenburg (1984, 1988) for the case  $n = 1$ . This generalization amounts on defining an “enclosed”  $(m+1)$ -volume of an  $m$ -dimensional immersed submanifold  $F : M^m \rightarrow \bar{M}^{m+n}$ ,  $m \geq 2$ , as the  $\Omega$ -volume defined by each one-parameter variation family  $F(x, t) = F_t(x)$  of  $F(x, 0) = F(x)$ , where  $\Omega$  is a semi-calibration on the ambient space  $\bar{M}$ , that is, an  $(m+1)$ -form  $\Omega$  which satisfies  $|\Omega(e_0, e_1, \dots, e_m)| \leq 1$ , for any orthonormal system  $e_i$  of  $T\bar{M}$ . A submanifold with calibrated extended tangent space  $H \oplus TM$  is a critical point of the functional area, for compactly supported  $\Omega$ -volume preserving variations, if and only if it has constant mean curvature  $\|H\|$ . In this case we have  $H = \|H\| \Omega \lrcorner \vec{S}$ . From a deeper inspection of this proof, one can see that the initial assumption of calibrated extended tangent space can be dropped, since it will appear as a consequence of being a critical point itself. This will be explained in detail in a future paper, and also its relations with Morgan’s formalism. Assuming that  $M$  has parallel mean curvature  $H$ , a second variation is then computed, and its non-negativeness defines stability of  $M$ . This

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corresponds to the non-negativeness of the quadratic form associated with the  $L^2$ -self-adjoint  $\Omega$ -Jacobi operator  $\mathcal{J}_\Omega(W) = \mathcal{J}(W) + m\|H\|C_\Omega(W)$ , acting on sections in the twisted normal bundle  $H_{0,T}^1(NM) = \mathcal{F} \oplus H_0^1(E)$ , where the set  $\mathcal{F}$  of  $H_0^1$ -functions with zero mean value is identified with the set of sections of the form  $f\nu$ , with  $f \in \mathcal{F}$  and  $\nu = H/\|H\|$ , and where  $E$  is the orthogonal complement of  $\nu$  in the normal bundle. This Jacobi operator is the usual one, but with an extra term, namely a multiple of a first order differential operator  $C_\Omega(W)$  that depends on  $\Omega$ . The twisted normal bundle is the  $H^1$ -completion of the vector space generated by the set  $\mathcal{F}_\Omega$  of compactly supported infinitesimal  $\Omega$ -volume preserving variations, and, in general, we do not know whether it is larger than  $\mathcal{F}_\Omega$  itself. Thus,  $\Omega$ -stability implies that the area functional of  $F_t$  decreases when  $t$  approaches  $t_0 = 0$ , for any family of  $\Omega$ -volume preserving variations  $F_t$  of  $F$ , but we do not know whether the converse also holds always. In case the ambient space is the Euclidean space  $\mathbb{R}^{m+n}$ , then a unit  $m$ -sphere of an  $\Omega$ -calibrated Euclidean subspace  $\mathbb{R}^{m+1}$  of  $\mathbb{R}^{m+n}$  is  $\Omega$ -stable if and only if, for any  $(n-1)$ -tuple of functions  $f_\alpha \in C^\infty(\mathbb{S}^m)$ ,  $2 \leq \alpha \leq n$ , the following integral inequality holds:

$$\sum_{\alpha < \beta} -2m \int_{\mathbb{S}^m} f_\alpha \xi(W_\alpha, W_\beta)(\nabla f_\beta) dM \leq \sum_{\alpha} \int_{\mathbb{S}^m} \|\nabla f_\alpha\|^2 dM, \quad (1)$$

where  $W_\alpha$  is a fixed global parallel orthonormal (o.n.) frame of  $\mathbb{R}^{n-1}$ , the orthogonal complement of  $\mathbb{R}^{m+1}$  spanned by  $\mathbb{S}^m$ , and  $\xi$  is the  $T^*\mathbb{S}^m$ -valued 2-form on  $\mathbb{R}_{/\mathbb{S}^m}^{n-1}$

$$\xi(W, W')(X) = \Omega(W, W', *X), \quad W, W' \in \mathbb{R}^{n-1}, X \in T^*\mathbb{S}^m$$

where  $*$  :  $T\mathbb{S}^m \rightarrow \wedge^{m-1}T\mathbb{S}^m$  is the star operator. If (1) holds and

$$\bar{\nabla}_W \Omega(W, e_1, \dots, e_m) = 0, \quad \forall W \in N\mathbb{S}^m, \quad (2)$$

where  $e_i$  is an o.n. frame of  $T\mathbb{S}^m$ , then in (Salavessa, 2010, proposition 4.5) we have shown that for each  $\alpha < \beta$ ,  $\xi(W_\alpha, W_\beta)$  must be co-exact as a 1-form on  $\mathbb{S}^m$ , that is,

$$\xi_{\alpha\beta} := \xi(W_\alpha, W_\beta) = \delta \omega_{\alpha\beta},$$

for some globally defined 2-form  $\omega_{\alpha\beta}$  on  $\mathbb{S}^m$ . This is the case when  $\Omega$  is a parallel  $(m+1)$ -form on  $\mathbb{R}^{m+n}$ . Using these forms  $\omega_{\alpha\beta}$ , the stability condition (1) is translated into the *long  $\Omega$ -Cauchy-Riemannian integral inequality*:

$$\sum_{\alpha < \beta} -2m \int_{\mathbb{S}^m} \omega_{\alpha\beta}(\nabla f_\alpha, \nabla f_\beta) dM \leq \sum_{\alpha} \int_{\mathbb{S}^m} \|\nabla f_\alpha\|^2 dM. \quad (3)$$

If we fix  $\alpha < \beta$ , and set  $f = f_\alpha$ ,  $h = f_\beta$ , and  $f_\gamma = 0 \forall \gamma \neq \alpha, \beta$ , (1) reduces to

$$-2m \int_{\mathbb{S}^m} f \xi_{\alpha\beta}(\nabla h) dM \leq \int_{\mathbb{S}^m} \|\nabla f\|^2 dM + \int_{\mathbb{S}^m} \|\nabla h\|^2 dM, \quad (4)$$

and if we replace  $f$  by  $cf$ , and  $h$  by  $c^{-1}h$ , where  $c^2 = \|\nabla h\|_{L^2}/\|\nabla f\|_{L^2}$ , then we obtain the corresponding equivalent *short  $\Omega$ -Cauchy-Riemannian, integral inequality*

$$-m \int_{\mathbb{S}^m} \omega_{\alpha\beta}(\nabla f, \nabla h) dM \leq \sqrt{\int_{\mathbb{S}^m} \|\nabla f\|^2 dM} \sqrt{\int_{\mathbb{S}^m} \|\nabla h\|^2 dM}, \quad (5)$$

holding for all functions  $f, h \in C^\infty(\mathbb{S}^m)$ .

The  $\Omega$ -stability of a submanifold with calibrated extended tangent space and parallel mean curvature depends on the curvature of the ambient space and on the calibration  $\Omega$  (Salavessa, 2010). It always holds on Euclidean spheres if  $C_\Omega$  vanish. This last condition is equivalent to the condition (2) and  $\xi \equiv 0$  ((Salavessa, 2010), Lemma 4.4). In the case  $n = 2$  the later condition is satisfied, but for  $n \geq 3$  the operator  $C_\Omega$  may not vanish for spheres, even if  $\Omega$  is parallel. If  $C_\Omega$  does not vanish, spheres of calibrated vector subspaces may not be  $\Omega$ -stable.

We first consider  $\Omega$  any parallel  $(m+1)$ -form on  $\mathbb{R}^{m+n}$ . Laplace spherical harmonics of  $\mathbb{S}^m$  of degree  $l$  are the eigenfunctions for the closed eigenvalue problem with respect to the Laplacian operator corresponding to the eigenvalue  $\lambda_l = l(l+m-1)$ , and they are just the harmonic homogeneous polynomial functions of degree  $l$  of  $\mathbb{R}^{m+1}$  restricted to  $\mathbb{S}^m$ . We denote by  $E_{\lambda_l}$  the finite-dimensional subspace of  $H^1(\mathbb{S}^m)$  spanned by these  $\lambda_l$ -eigenfunctions. In the first theorem we show how each 1-form  $\xi_{\alpha\beta}$  transforms a spherical harmonic  $f$  into another spherical harmonic  $h$ :

**Theorem 1.1.** *If  $\Omega$  is parallel, then for each  $f \in E_{\lambda_l}$ ,  $h = \xi_{\alpha\beta}(\nabla f)$  is also in  $E_{\lambda_l}$ , and it is  $L^2$ -orthogonal to  $f$ .*

In this paper we study the stability of the unit 3-sphere of a 2-dimensional complex subspace of  $\mathbb{C}^3$  with respect to the Kähler calibration. In this case  $C_\Omega$  does not vanish. Let  $\varpi$  be the Kähler form of  $\mathbb{C}^3 = \mathbb{R}^6$ , and  $\Omega$  the Kähler calibration of rank 4,

$$\varpi = dx^{12} + dx^{34} + dx^{56}, \quad \Omega = \frac{1}{2}\varpi^2.$$

The unit sphere of  $\mathbb{R}^4 \times \{0\}$  is immersed into  $\mathbb{R}^6 = \mathbb{C}^3$ , by the inclusion map  $\phi = (\phi_1, \dots, \phi_4, 0) : \mathbb{S}^3 \rightarrow \mathbb{C}^3$ . We have only one of those 1-forms

$$\xi := \xi_{56} = *(d\phi^1 \wedge d\phi^2 + d\phi^3 \wedge d\phi^4) = \phi^1 d\phi^2 - \phi^2 d\phi^1 + \phi^3 d\phi^4 - \phi^4 d\phi^3,$$

and  $\xi = \delta\omega$ , with  $\omega = \frac{1}{2}*\xi = \frac{1}{2}(d\phi^1 \wedge d\phi^2 + d\phi^3 \wedge d\phi^4) = \frac{1}{2}\phi^*\varpi$ . Our main theorem is the following:

**Theorem 1.2.** *Three-dimensional spheres of  $\mathbb{C}^2$  are  $\Omega$ -stable submanifolds of  $\mathbb{C}^3$  with parallel mean curvature, where  $\Omega = \frac{1}{2}\varpi^2$  is the Kähler calibration of rank 4.*

The Cauchy-Riemann inequality version of the  $\Omega$ -stability is described in the corollary:

**Corollary 1.1.** *The Cauchy-Riemann inequality*

$$-\int_{\mathbb{S}^3} \varpi(\nabla f, \nabla h) dM \leq \frac{2}{3} \sqrt{\int_{\mathbb{S}^3} \|\nabla f\|^2 dM} \sqrt{\int_{\mathbb{S}^3} \|\nabla h\|^2 dM}$$

holds for any smooth functions  $f$  and  $h$  of  $\mathbb{S}^3$ , with equality if and only if  $f, h \in E_{\lambda_1}$ , with  $f = \sum_i \mu_i \phi_i$  and  $h = \sum_i \sigma_i \phi_i$ , where  $\sigma_2 = -\mu_1$ ,  $\sigma_1 = \mu_2$ ,  $\sigma_4 = -\mu_3$ ,  $\sigma_3 = \mu_4$ .

Finally, we state that the 3-sphere is the unique smooth closed submanifold that solves the  $\Omega$ -isoperimetric problem among a certain class of immersed submanifolds:

**Theorem 1.3.** *The unit 3-sphere of a complex 2-dimensional subspace of  $\mathbb{C}^3$  is the unique closed immersed 3-dimensional submanifold  $\phi : M \rightarrow \mathbb{C}^3$  with parallel mean curvature, trivial normal bundle, and complex extended tangent space  $H \oplus TM$ , that is  $\Omega$ -stable for the Kähler calibration of rank 4, and satisfies the inequality*

$$\int_M S(2 + h\|H\|)dM \leq 0,$$

where  $h$  and  $S$  are the height functions  $h = \langle \phi, \nu \rangle$  and  $S = \sum_{ij} \langle \phi, (B(e_i, e_j))^F \rangle B^V(e_i, e_j)$ .

**Remark.** On a closed Kähler manifold  $(M, J)$  with Kähler form  $\varpi(X, Y) = g(JX, Y)$ , if  $f, h : M \rightarrow \mathbb{R}$  are smooth functions, then by the Cauchy-Schwarz inequality,

$$\left| \int_M \varpi(\nabla f, \nabla h) dM \right| \leq \sqrt{\int_M \|\nabla f\|^2 dM} \sqrt{\int_M \|\nabla h\|^2 dM},$$

with equality if and only if  $\nabla h = \pm J \nabla f$ , or equivalently  $f \pm ih : M \rightarrow \mathbb{C}$  is a holomorphic function. If this is the case, then  $f$  and  $h$  are constant functions. On the other hand, globally defined functions, sufficiently close to holomorphic functions defined on a sufficiently large open set, are expected to satisfy an almost equality. This is not the case of  $\mathbb{S}^3$ , which is not a complex manifold, and somehow explains the coefficient  $2/3$  in Corollary 1.1.

**Remark.** In the case of 3-spheres in  $\mathbb{C}^3$  we have only one form  $\xi_{\alpha\beta}$ , that is, the long Cauchy-Riemann inequality is the short one. We wonder if a general proof of short Cauchy-Riemann inequalities can be allways obtained for Euclidean  $m$ -spheres on  $\mathbb{R}^{m+n}$ , by using the spectral theory of spheres, when  $\Omega$  is any parallel calibration. Note that (4) is immediately satisfied for  $f, h \in E_{\lambda_l}$ , if  $\lambda_l \geq m^2$ , that is  $l \geq m$ , so it remains to consider the cases  $l \leq m - 1$ . For 3-spheres we have to consider polynomial functions up to order  $l = 2$ , while for 2-spheres we have to consider only the case  $l = 1$ . A related remark is given in the end of section 3.

## 2. Preliminaries

We consider an oriented Riemannian manifold  $M$  of dimension  $m$ , with Levi-Civita connection  $\nabla$  and Ricci tensor  $\text{Ricci}^M : TM \rightarrow TM$ . In what follows  $e_1, \dots, e_m$  denotes a local direct o.n. frame.

**Lemma 2.1.** *Let  $\xi$  be a co-exact 1-form on a Riemannian manifold  $M$ , with  $\xi = \delta\omega$ , where  $\omega$  is a 2-form. Then for any function  $f \in C^2(M)$ ,*

$$\xi(\nabla f) = \text{div}(\nabla^\omega f),$$

where  $\nabla^\omega f = \sum_i \omega(\nabla f, e_i) e_i$ . Moreover, for any  $f, h \in C_0^\infty(M)$

$$\int_M f \xi(\nabla h) dM = \int_M \omega(\nabla f, \nabla h) dM = - \int_M h \xi(\nabla f) dM.$$

**Proof.** We may assume at a point  $x_0$ ,  $\nabla e_i = 0$ . Then at  $x_0$

$$\begin{aligned}\xi(\nabla f) &= \delta\omega(\nabla f) = -\sum_i \nabla_{e_i} \omega(e_i, \nabla f) = \sum_i -\nabla_{e_i}(\omega(e_i, \nabla f)) + \omega(e_i, \nabla_{e_i} \nabla f) \\ &= \operatorname{div}(\nabla^\omega f) + \sum_{ij} \operatorname{Hess} f(e_i, e_j) \omega(e_i, e_j).\end{aligned}$$

The last equality proves the first equality of the lemma, because  $\operatorname{Hess} f(e_i, e_j)$  is symmetric on  $i, j$  and  $\omega(e_i, e_j)$  is skew-symmetric. The other equalities of the lemma follow from  $\operatorname{div}(fX) = \langle \nabla f, X \rangle + f \operatorname{div}(X)$ , holding for any vector field  $X$  and function  $f$ .  $\square$

The  $\delta$  and star operators acting on  $p$ -forms on an oriented Riemannian  $m$ -manifold  $M$  satisfy  $\delta = (-1)^{mp+m+1} * d *$ ,  $** = (-1)^{p(m-p)} Id$ , and for a 1-form  $\xi$  the DeRham Laplacian  $\Delta$  and the rough Laplacian  $\bar{\Delta}$  are related by the following formulas

$$\begin{aligned}\Delta \xi(X) &= (d\delta + \delta d)\xi(X) = -\bar{\Delta} \xi(X) + \xi(\operatorname{Ricci}^M(X)), \\ \bar{\Delta} \xi(X) &= \operatorname{trace} \nabla^2 \xi(X) = \sum_i \nabla_{e_i} \nabla_{e_i} \xi(X) - \nabla_{\nabla_{e_i} e_i} \xi(X).\end{aligned}$$

If  $\xi = \delta\omega$ , then  $\delta\xi = 0$ , and so  $\Delta \xi(X) = \delta d\xi(X) = -\sum_i \nabla_{e_i} (d\xi)(e_i, X)$ . We also recall the following well-known formula (see e.g. Salavessa & Pereira do Vale (2006)) for  $f \in C^\infty(M)$ ,

$$(\bar{\Delta} df)(X) = \sum_i \nabla_{e_i, e_i}^2 df(X) = g(\nabla(\Delta f), X) + df(\operatorname{Ricci}^M(X)).$$

Thus,

$$\begin{aligned}\bar{\Delta}(\nabla f) &= \nabla(\Delta f) + \operatorname{Ricci}^M(\nabla f), \\ (\bar{\Delta} \xi)(\nabla f) &= -(\delta d\xi)(\nabla f) + \xi(\operatorname{Ricci}^M(\nabla f)).\end{aligned}\tag{6}$$

Now we suppose that  $M$  is an immersed oriented hypersurface of a Riemannian manifold  $M'$ , with Riemannian metric  $\langle, \rangle$ , defined by an immersion  $\phi : M \rightarrow M'$  with unit normal  $\nu$ , second fundamental form  $B$  and corresponding Weingarten operator  $A$  in the  $\nu$  direction, given by

$$B(e_i, e_j) = \langle A(e_i), e_j \rangle = \langle \nabla'_{e_i} e_j, \nu \rangle = -\langle e_j, \nabla'_{e_i} \nu \rangle,$$

where  $\nabla'$  denotes the Levi-Civita connection on  $M'$ . The scalar mean curvature of  $M$  is given by

$$H = \frac{1}{m} \operatorname{Trace} B = \sum_i \frac{1}{m} B(e_i, e_i).$$

The curvature operator of  $M'$ ,  $R'(X, Y, Z, W) = \langle -\nabla'_X \nabla'_Y Z + \nabla'_Y \nabla'_X Z + \nabla'_{[X, Y]} Z, W \rangle$ , can be seen as a self-adjoint operator of wedge bundles  $R' : \wedge^2 TM' \rightarrow \wedge^2 TM'$ ,

$$\langle R'(u \wedge v), z \wedge w \rangle = R'(u, v, z, w),$$

and so  $R'(u \wedge v) = \sum_{i < j} R'(u, v, e_i, e_j) e_i \wedge e_j$ , where

$$\langle u \wedge v, z \wedge w \rangle = \det \begin{bmatrix} \langle u, z \rangle & \langle u, w \rangle \\ \langle v, z \rangle & \langle v, w \rangle \end{bmatrix}.$$

In what follows, we suppose that  $\hat{\xi}$  is a parallel  $(m-1)$ -form on  $M'$ , and  $\xi$  is given by

$$\xi = * \phi^* \hat{\xi}$$

where  $*$  is the star operator on  $M$ . In this case  $\xi$  is obviously co-closed, but not necessarily co-exact. We employ the usual inner products in  $p$ -forms and morphisms.

**Lemma 2.2.** *Assume  $m \geq 3$ . Then for all  $i, j$*

$$\begin{aligned} (\nabla_{e_i} \xi)(e_j) &= \sum_k -B(e_i, e_k) \hat{\xi}(v, *(e_k \wedge e_j)) = -\hat{\xi}(v, *(A(e_i) \wedge e_j)), \\ \Delta \xi(e_j) &= \delta d\xi(e_j) = \hat{\xi} \left( v, *(e_j \wedge (m \nabla H - [\text{Ricci}^{M'}(v)]^T)) + R'(e_j \wedge v) \right) + \xi(\Theta_B(e_j)), \end{aligned}$$

where  $[\text{Ricci}^{M'}(v)]^T = \sum_k \text{Ricci}^{M'}(v, e_k) e_k$  and  $\Theta_B : TM \rightarrow TM$  is the morphism given by,  $\Theta_B = \|B\|^2 Id + mHA - 2A^2$ .

**Proof.** We fix a point  $x_0 \in M$  and take  $e_i$  a local o.n. frame s.t.  $\nabla e_i(x_0) = 0$ . We will compute  $d\xi(e_i, e_j)$ , at  $x$  on a neighbourhood of  $x_0$ . Recall that for any  $p$ -form  $\sigma$ , we have  $*\sigma = \sigma*$ , where the star operator on the r.h.s. can be seen as acting on  $\wedge^{m-p} TM$ , with  $*e_i = (-1)^{i-1} e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge e_m$ , and for  $i < j$ ,  $*(e_i \wedge e_j) = (-1)^{i+j-1} e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge e_m$ . Using the fact that  $\hat{\xi}$  is a parallel form on  $M'$ , we have for  $x$  near  $x_0$ ,

$$\begin{aligned} \nabla_{e_i}(\xi(e_j)) &= \sum_{k \neq j} (-1)^{j-1} \hat{\xi}(e_1, \dots, \nabla'_{e_i} e_k, \dots, \hat{e}_j, \dots, e_m) \\ &= \sum_{k < j} (-1)^{k+j} \hat{\xi}(\nabla'_{e_i} e_k, e_1, \dots, \hat{e}_k, \dots, \hat{e}_j, \dots, e_m) \\ &\quad + \sum_{k > j} (-1)^{k+j-1} \hat{\xi}(\nabla'_{e_i} e_k, e_1, \dots, \hat{e}_j, \dots, \hat{e}_k, \dots, e_m) \\ &= \sum_{k < j} -\langle \nabla_{e_i} e_k, e_j \rangle \hat{\xi}(*e_k) - B(e_i, e_k) \hat{\xi}(v, *(e_k \wedge e_j)) \\ &\quad + \sum_{k > j} -\langle \nabla_{e_i} e_k, e_j \rangle \hat{\xi}(*e_k) + B(e_i, e_k) \hat{\xi}(v, *(e_j \wedge e_k)) \\ &= \xi(\nabla_{e_i} e_j) + \sum_{k \neq j} -B(e_i, e_k) \hat{\xi}(v, *(e_k \wedge e_j)). \end{aligned}$$

Hence,  $(\nabla_{e_i} \xi)(e_j) = \sum_{k \neq j} -B(e_i, e_k) \hat{\xi}(v, *(e_k \wedge e_j))$ , which proves the first sequence of equalities of the lemma. Now,

$$\begin{aligned} d\xi(e_i, e_j) &= (\nabla_{e_i} \xi)(e_j) - (\nabla_{e_j} \xi)(e_i) \\ &= \sum_{k \neq j} -B(e_i, e_k) \hat{\xi}(v, *(e_k \wedge e_j)) + \sum_{k \neq i} B(e_j, e_k) \hat{\xi}(v, *(e_k \wedge e_i)), \end{aligned}$$

and by Codazzi's equation,

$$\begin{aligned} (\nabla_{e_i} B)(e_j, e_k) &= (\nabla_{e_j} B)(e_i, e_k) - R'(e_i, e_j, e_k, v) \\ \sum_i (\nabla_{e_i} B)(e_i, e_k) &= m \nabla_{e_k} H - \text{Ricci}^{M'}(e_k, v). \end{aligned}$$

Note that  $B_{ik} = (\nabla_{e_j} B)(e_i, e_k)$  is a symmetric matrix, and if we define  $A_{ki} = \hat{\xi}(v, *(e_k \wedge e_i))$  (valuing zero if  $k = i$ ), then  $A_{ik}$  is skew-symmetric. Thus,  $\sum_{k \neq i} B_{ik} A_{ki} = \sum_{k,i} B_{ik} A_{ki} = 0$ . Furthermore, if we set  $C_{ik} = -R'(e_i, e_j, e_k, v)$ , then  $C_{ik} - C_{ki} = R'(e_k, e_i, e_j, v)$ . Hence,

$$\sum_i \sum_{k \neq i} C_{ik} A_{ki} = \sum_{ik} C_{ik} A_{ki} = \sum_{ik} \frac{1}{2} ((C_{ik} + C_{ki}) + (C_{ik} - C_{ki})) A_{ki} = \sum_{ki} \frac{1}{2} R'(e_k, e_i, e_j, v) A_{ki}.$$

Therefore, for each  $j$ , at  $x_0$

$$\begin{aligned} -\delta d\xi(e_j) &= \sum_i \nabla_{e_i} (d\xi(e_i, e_j)) \\ &= \sum_{k \neq j} \sum_i -(\nabla_{e_i} B)(e_i, e_k) \hat{\xi}(v, *(e_k \wedge e_j)) - B(e_i, e_k) \nabla_{e_i} (\hat{\xi}(v, *(e_k \wedge e_j))) \\ &\quad + \sum_{k \neq i} \sum_j (\nabla_{e_i} B)(e_j, e_k) \hat{\xi}(v, *(e_k \wedge e_i)) + B(e_j, e_k) \nabla_{e_i} (\hat{\xi}(v, *(e_k \wedge e_i))) \\ &= \sum_{k \neq j} (-m \nabla_{e_k} H + Ricci^M(e_k, v)) \hat{\xi}(v, *(e_k \wedge e_j)) + \sum_{k,i} \frac{1}{2} R'(e_k, e_i, e_j, v) \hat{\xi}(v, *(e_k \wedge e_i)) + S \end{aligned}$$

where

$$\begin{aligned} S &= \sum_i \sum_{k < j} (-1)^{k+j} B(e_i, e_k) \hat{\xi}(\nabla'_{e_i} v, e_1, \dots, \hat{e}_k, \dots, \hat{e}_j, \dots, e_m) \\ &\quad + \sum_i \sum_{k > j} (-1)^{k+j-1} B(e_i, e_k) \hat{\xi}(\nabla'_{e_i} v, e_1, \dots, \hat{e}_j, \dots, \hat{e}_k, \dots, e_m) \\ &\quad + \sum_i \sum_{k < i} (-1)^{k+i-1} B(e_j, e_k) \hat{\xi}(\nabla'_{e_i} v, e_1, \dots, \hat{e}_k, \dots, \hat{e}_i, \dots, e_m) \\ &\quad + \sum_i \sum_{k > i} (-1)^{k+i} B(e_j, e_k) \hat{\xi}(\nabla'_{e_i} v, e_1, \dots, \hat{e}_i, \dots, \hat{e}_k, \dots, e_m) \\ &= \sum_i \sum_{k < j} -B(e_i, e_k) B(e_i, e_k) \xi(e_j) + B(e_i, e_j) B(e_i, e_k) \xi(e_k) \\ &\quad + \sum_i \sum_{k > j} B(e_i, e_j) B(e_i, e_k) \xi(e_k) - B(e_i, e_k) B(e_i, e_k) \xi(e_j) \\ &\quad + \sum_i \sum_{k < i} B(e_i, e_k) B(e_j, e_k) \xi(e_i) - B(e_i, e_i) B(e_j, e_k) \xi(e_k) \\ &\quad + \sum_i \sum_{k > i} -B(e_i, e_i) B(e_j, e_k) \xi(e_k) + B(e_i, e_k) B(e_j, e_k) \xi(e_i). \end{aligned}$$

At this point we may assume that at  $x_0$  the basis  $e_i$  diagonalizes the second fundamental form, that is,  $B(e_i, e_j) = \lambda_i \delta_{ij}$ . Then,

$$\begin{aligned} S &= \sum_i \sum_{k < j} -\delta_{ik} \lambda_i^2 \xi(e_j) + \delta_{ij} \delta_{ik} \lambda_i^2 \xi(e_k) + \sum_i \sum_{k > j} \delta_{ij} \delta_{ik} \lambda_i^2 \xi(e_k) - \delta_{ik} \lambda_i^2 \xi(e_j) \\ &\quad + \sum_i \sum_{k < i} \delta_{ik} \delta_{jk} \lambda_k^2 \xi(e_i) - \delta_{ii} \delta_{jk} \lambda_i \lambda_j \xi(e_k) + \sum_i \sum_{k > i} -\delta_{ii} \delta_{jk} \lambda_i \lambda_j \xi(e_k) + \delta_{ik} \delta_{jk} \lambda_k^2 \xi(e_i) \\ &= \sum_{i < j} -\lambda_i^2 \xi(e_j) + \sum_{i > j} -\lambda_i^2 \xi(e_j) + \sum_{j < i} -\lambda_i \lambda_j \xi(e_j) + \sum_{j > i} -\lambda_i \lambda_j \xi(e_j) \\ &= \sum_{i \neq j} -\lambda_i^2 \xi(e_j) - \lambda_i \lambda_j \xi(e_j) = \sum_i -\lambda_i^2 \xi(e_j) - \lambda_i \lambda_j \xi(e_j) + (\lambda_j^2 + \lambda_j^2) \xi(e_j) \\ &= -\|B\|^2 \xi(e_j) - mH \xi(A(e_j)) + 2\xi(A^2(e_j)), \end{aligned}$$

and the second sequence of equalities of the lemma is proved.  $\square$

If we suppose that  $\Theta_B = \mu(x)Id$ , taking  $e_i$  a diagonalizing o.n. basis of the second fundamental form,  $B(e_i, e_j) = \lambda_i \delta_{ij}$ , then each  $\lambda_i$  satisfies the quadratic equation

$$2\lambda_i^2 - mH\lambda_i + (\mu - \|B\|^2) = 0,$$

which implies that we have at most two distinct possible principal curvatures  $\lambda_{\pm}$ . Moreover, from the above equation, summing over  $i$ , we derive that  $\mu(x)$  must satisfy  $\mu(x) = \frac{m-2}{m}\|B\|^2 + mH^2$ , and so

$$\lambda_{\pm} = \frac{1}{4} \left( mH \pm \sqrt{\frac{16}{m}\|B\|^2 + m(m-8)H^2} \right).$$

Note that, from  $\|B\|^2 \geq m\|H\|^2$ , we have  $\frac{16}{m}\|B\|^2 + m(m-8)H^2 \geq (m-4)^2H^2$ , and so there are one or two distinct principal curvatures. If  $M$  is totally umbilical, then  $\|B\|^2 = mH^2$  and  $\mu = 2(m-1)\|H\|^2$ . The previous lemma leads to the following conclusion:

**Lemma 2.3.** *Assuming  $M' = \mathbb{R}^{m+1}$ ,  $m \geq 3$ , and taking  $M$  a hypersurface with constant mean curvature, with  $\Theta_B = \mu(x)Id$ , where  $\mu(x)$  is a smooth function on  $M$ , we get  $\mu(x) = \frac{m-2}{m}\|B\|^2 + mH^2$  and*

$$\Delta\xi = \mu\xi.$$

Furthermore,  $\xi$  is an eigenform for the DeRham Laplacian operator, that is  $\mu(x)$  is constant, if and only if  $\|B\|$  is constant.

In case  $M$  is a unit  $m$ -sphere  $\mathbb{S}^m$ , then  $\Theta_B = \mu Id$ , with  $\mu = 2(m-1)$ , and taking  $v_x = -x$  as unit normal, then, at each  $x \in \mathbb{S}^m$ ,

$$\begin{aligned} (\nabla_{e_i}\xi)(e_j) &= \hat{\xi}(x, *(e_i \wedge e_j)) \\ d\xi(e_i, e_j) &= 2\hat{\xi}(x, *(e_i \wedge e_j)) \\ \Delta\xi &= \delta d\xi = 2(m-1)\xi. \end{aligned}$$

**Lemma 2.4.** *If  $f \in C^\infty(\mathbb{S}^m)$ , then  $\Delta(\xi(\nabla f)) = \xi(\nabla \Delta f)$ .*

**Proof.** We fix a point  $x_0 \in \mathbb{S}^m$  and take  $e_i$  a local o.n. frame of the sphere s.t.  $\nabla_{e_i}(x_0) = 0$ . Let  $f \in C^\infty(\mathbb{S}^m)$ . The following computations are at  $x_0$ . Using the above formulas (6) and previous lemma, we have

$$\begin{aligned} \Delta(\xi(\nabla f)) &= \sum_i \nabla_{e_i}(\nabla_{e_i}(\xi(\nabla f))) = \sum_i \nabla_{e_i}((\nabla_{e_i}\xi)(\nabla f) + \xi(\nabla_{e_i}\nabla f)) \\ &= (\Delta\xi)(\nabla f) + 2(\nabla_{e_i}\xi)(\nabla_{e_i}\nabla f) + \xi(\nabla_{e_i}\nabla_{e_i}\nabla f) \\ &= -2(m-1)\xi(\nabla f) + \xi(\nabla \Delta f) + 2(m-1)\xi(\nabla f) + \sum_i 2(\nabla_{e_i}\xi)(\nabla_{e_i}\nabla f). \end{aligned}$$

Since  $Hess f(e_i, e_j)$  is symmetric in  $ij$  and by Lemma 2.3,  $(\nabla_{e_i}\xi)(e_j)$  is skew-symmetric, we have

$$\sum_i (\nabla_{e_i}\xi)(\nabla_{e_i}\nabla f) = \sum_{ij} Hess f(e_i, e_j)(\nabla_{e_i}\xi)(e_j) = 0,$$

and the lemma is proved.  $\square$

### 3. Proof of Theorem 1.1.

We denote by  $\nabla$  the Levi-Civita connection of  $\mathbb{S}^m$  induced by the flat connection  $\bar{\nabla}$  of  $\mathbb{R}^{m+n}$ . We are considering a parallel calibration  $\Omega$  on  $\mathbb{R}^{m+n}$ . We fix  $\alpha < \beta$  and define the 1-form on  $\mathbb{S}^m$

$$\xi = \xi(W_\alpha, W_\beta) = *\phi^*\hat{\xi} = \delta\omega,$$



where  $\hat{\xi} = \hat{\xi}_{\alpha\beta}$  and  $\omega = \omega_{\alpha\beta}$ .

We recall that the eigenvalues of  $\mathbb{S}^m$  for the closed Dirichlet problem are given by  $\lambda_l = l(l+m-1)$ , with  $l = 0, 1, 2, \dots$ . We denote by  $E_{\lambda_l}$  the eigenspace of dimension  $m_l$  corresponding to the eigenvalue  $\lambda_l$ , and by  $E_{\lambda_l}^+$  the  $L^2$ -orthogonal complement of the sum of the eigenspaces  $E_{\lambda_i}$ ,  $i = 1, \dots, l-1$ , and so it is the sum of all eigenspaces  $E_{\lambda}$  with  $\lambda \geq \lambda_l$ . If  $f \in E_{\lambda_l}$ , and  $h \in E_{\lambda_s}$ , then

$$\int_{\mathbb{S}^m} fh dM = 0 \text{ if } l \neq s \quad \text{and} \quad \int_{\mathbb{S}^m} \langle \nabla f, \nabla h \rangle dM = \delta_{ls} \lambda_l \int_{\mathbb{S}^m} fh dM.$$

There exists an  $L^2$ -orthonormal basis  $\psi_{l,\sigma}$  of  $L^2(\mathbb{S}^m)$  of eigenfunctions ( $1 \leq \sigma \leq m_l$ ). The Rayleigh characterization of  $\lambda_l$  is given by

$$\lambda_l = \inf_{f \in E_{\lambda_l}^+} \frac{\int_{\mathbb{S}^m} \|\nabla f\|^2 dM}{\int_{\mathbb{S}^m} f^2 dM},$$

and the infimum is attained for  $f \in E_{\lambda_l}$ . Each eigenspace  $E_{\lambda_l}$  is exactly composed by the restriction to  $\mathbb{S}^m$  of the harmonic homogeneous polynomial functions of degree  $l$  of  $\mathbb{R}^{m+1}$ , and it has dimension  $m_l = \binom{m+l}{m} - \binom{m+l-2}{m}$ . Thus, each eigenfunction  $\psi \in E_{\lambda_l}$  is of the form  $\psi = \sum_{|a|=l} \mu_a \phi^a$ , where  $\mu_a$  are some scalars and  $a = (a_1, \dots, a_{m+1})$  denotes a multi-index of length  $|a| = a_1 + \dots + a_{m+1} = l$  and

$$\phi^a = \phi_1^{a_1} \cdot \dots \cdot \phi_{m+1}^{a_{m+1}}.$$

From  $\nabla \phi_i = \varepsilon_i^\top$  and  $\sum_i \phi_i^2 = 1$ , we see that

$$\begin{cases} \langle \nabla \phi_i, \nabla \phi_j \rangle = \delta_{ij} - \phi_i \phi_j & \|\nabla \phi_i\|^2 = 1 - \phi_i^2 \\ \int_{\mathbb{S}^m} \phi_i^2 dM = \frac{1}{m+1} |\mathbb{S}^m| & \int_{\mathbb{S}^m} \|\nabla \phi_i\|^2 dM = \lambda_1 \int_{\mathbb{S}^2} \phi_i^2 dM = \frac{m}{m+1} |\mathbb{S}^m|. \end{cases} \quad (7)$$

We also denote by  $\int_{\mathbb{S}^m} \phi^2 dM$  any of the integrals  $\int_{\mathbb{S}^m} \phi_i^2 dM$ ,  $i = 1, \dots, m+1$ . We recall the following:

**Lemma 3.1.** *If  $P : \mathbb{S}^m \rightarrow \mathbb{R}$  is a homogeneous polynomial function of degree  $l$ , then*

$$\int_{\mathbb{S}^m} P(x) dM = \frac{1}{\lambda_l} \int_{\mathbb{S}^m} \Delta^0 P(x) dM.$$

*In particular,*

$$\int_{\mathbb{S}^m} \phi^a dM = \sum_{1 \leq i \leq m+1} \frac{a_i(a_i-1)}{l(l+m-1)} \int_{\mathbb{S}^m} \phi^{a-2\varepsilon_i} dM,$$

*where the terms  $a_i < 2$  are considered to vanish. Thus, if some  $a_i$  is odd this integral vanishes.*

**Proof of Theorem 1.1.** By Lemma 2.4, if  $f \in E_{\lambda_k}$  then  $\xi(\nabla f) \in E_{\lambda_k}$ . From

$$\int_{\mathbb{S}^m} f \xi(\nabla f) dM = \int_{\mathbb{S}^m} \omega(\nabla f, \nabla f) dM = 0$$

we conclude that  $f$  and  $h = \xi(\nabla f)$  are  $L^2$ -orthogonal. □

**Remark.** Let us consider  $f, h \in E_{\lambda_l}$ , and take the globally defined vector field of  $\mathbb{S}^m$ ,  $\xi^\sharp = \sum_j \xi(e_j)e_j$ . From Lemma 2.2, we have

$$\langle \nabla h, \nabla(\xi(\nabla f)) \rangle = -\hat{\xi}(v, *(\nabla h \wedge \nabla f)) + \text{Hess}f(\nabla h, \xi^\sharp).$$

By Theorem 1.1,  $\xi(\nabla f) \in E_{\lambda_l}$  as well. The term  $\text{Hess}f(\nabla h, \xi^\sharp)$  is a sum of polynomial functions of degree  $2l - 3 + k_\xi$  where  $k_\xi$  depends on  $\xi^\sharp$ , when expressed in terms of  $\phi^i$ . Let us suppose that all  $k_\xi$  are even. Then by Lemma 3.1,  $\int_{\mathbb{S}^m} \text{Hess}f(\nabla h, \xi^\sharp) dM = 0$ . Since  $\lambda_l \geq m$ , and taking into consideration that  $\Omega$  is a semi-calibration,

$$\begin{aligned} - \int_{\mathbb{S}^m} h \xi(\nabla f) dM &= - \frac{1}{\lambda_l} \int_{\mathbb{S}^m} \langle \nabla h, \nabla(\xi(\nabla f)) \rangle dM = \frac{1}{\lambda_l} \int_{\mathbb{S}^m} \hat{\xi}(v, *(\nabla h \wedge \nabla f)) dM \\ &\leq \frac{1}{\lambda_l} \int_{\mathbb{S}^m} \|\nabla h\| \|\nabla f\| dM \leq \frac{1}{m} \|\nabla f\|_{L^2} \|\nabla h\|_{L^2}. \end{aligned}$$

Thus, in this case the short Cauchy-Riemann inequality holds. Inspection of  $\xi$  must be required for each case of  $\Omega$ . A general proof of the short Cauchy-Riemann integral inequality, under appropriate conditions on  $\Omega$ , will be developed in a future paper.

#### 4. 3-spheres of $\mathbb{C}^2$ in $\mathbb{C}^3$

In this section we specialize the Cauchy-Riemann inequalities for the case  $m = n = 3$  and for  $\mathbb{R}^6 = \mathbb{C}^3$  we will consider the Kähler calibration  $\frac{1}{2}\bar{\omega}^2$  that calibrates the complex two-dimensional subspaces, that is,

$$\Omega = dx^{1234} + dx^{1256} + dx^{3456}.$$

Thus, fixing  $W_5 = \varepsilon_5$  and  $W_6 = \varepsilon_6$  we have  $\hat{\xi} := \hat{\xi}_{56} = dx^{12} + dx^{34}$ , and

$$\xi := \xi_{56} = *\phi^*\hat{\xi} = *(d\phi^{12} + d\phi^{34}).$$

The volume element of  $\mathbb{S}^m$  is  $\text{Vol}_{\mathbb{S}^m} = \sum_i (-1)^{i-1} \phi_i d\phi^1 \dots \hat{\phi}_i \dots d\phi^m$ , and  $*\xi$  is the unique 2-form s.t.  $\xi \wedge *\xi = \|\xi\|^2 \text{Vol}_{\mathbb{S}^m}$ . Using (7) we see that  $\|\xi\| = \|\phi^*\hat{\xi}\| = 1$ . Hence

$$\begin{aligned} \xi &= \phi_1 d\phi^2 - \phi_2 d\phi^1 + \phi_3 d\phi^4 - \phi_4 d\phi^3 \\ *\xi &= d\phi^1 \wedge d\phi^2 + d\phi^3 \wedge d\phi^4 = \frac{1}{2} d\xi =: d*\omega. \end{aligned}$$

Therefore, we may take  $*\omega = \frac{1}{2}\xi$ , that is

$$\omega = \frac{1}{2} *\xi = \frac{1}{2} (d\phi^1 \wedge d\phi^2 + d\phi^3 \wedge d\phi^4) = \frac{1}{2} \phi^* \bar{\omega}.$$

Hence, to prove Theorem 1.2 and Corollary 1.1 we have to verify that, for any functions  $f, h \in C^\infty(\mathbb{S}^3)$ , one of the following equivalent inequalities holds:

$$\begin{aligned} \int_{\mathbb{S}^3} -3\omega(\nabla f, \nabla h) dM &= \int_{\mathbb{S}^3} -3f\xi(\nabla h) dM \leq \|\nabla f\|_{L^2} \|\nabla h\|_{L^2} \\ \int_{\mathbb{S}^3} -6\omega(\nabla f, \nabla h) dM &= \int_{\mathbb{S}^3} -6f\xi(\nabla h) dM \leq \|\nabla f\|_{L^2}^2 + \|\nabla h\|_{L^2}^2. \end{aligned} \tag{8}$$

By Theorem 1.1 we only need to consider both  $f, h \in E_{\lambda_l}$ , for some  $l$ . Note that  $\lambda_3 = 15$  and since  $\Omega$  is a calibration,  $\|\xi(X)\| \leq \|X\|$ .

**Lemma 4.1.** *If  $f, h \in E_{\lambda_3}^+$  are nonzero, (8) holds, with strict inequality.*

**Proof.** By Schwartz inequality and Rayleigh characterization

$$\int_{\mathbb{S}^3} -3f\xi(\nabla h)dM \leq 3\|f\|_{L^2}\|\nabla h\|_{L^2} \leq \frac{3}{\sqrt{\lambda_3}}\|\nabla f\|_{L^2}\|\nabla h\|_{L^2} < \|\nabla f\|_{L^2}\|\nabla h\|_{L^2},$$

with strict inequality in the last one, since neither  $f$  nor  $h$  may be constant.  $\square$

We now verify that (8) holds for  $f, h \in E_{\lambda_1}$  and  $f, h \in E_{\lambda_2}$ . From (7) and Lemma 3.1, we have for  $i \neq j$

$$\begin{aligned} \int_{\mathbb{S}^3} \phi^2 dM &= \frac{1}{4}|\mathbb{S}^3|, & \int_{\mathbb{S}^3} \phi_i^2 \phi_j^2 dM &= \frac{1}{6} \int_{\mathbb{S}^3} \phi^2 dM \\ \int_{\mathbb{S}^3} \phi^4 dM &= \frac{1}{2} \int_{\mathbb{S}^3} \phi^2 dM, & \int_{\mathbb{S}^3} \|\nabla \phi\|^2 dM &= 3 \int_{\mathbb{S}^3} \phi^2 dM \\ \omega(\nabla \phi_1, \nabla \phi_2) &= \frac{1}{2}(1 - \phi_1^2 - \phi_2^2) & \omega(\nabla \phi_1, \nabla \phi_3) &= \frac{1}{2}(-\phi_2 \phi_3 + \phi_1 \phi_4) \\ \omega(\nabla \phi_1, \nabla \phi_4) &= \frac{1}{2}(-\phi_2 \phi_4 - \phi_1 \phi_3) & \omega(\nabla \phi_2, \nabla \phi_3) &= \frac{1}{2}(\phi_1 \phi_3 + \phi_4 \phi_2) \\ \omega(\nabla \phi_2, \nabla \phi_4) &= \frac{1}{2}(\phi_1 \phi_4 - \phi_2 \phi_3) & \omega(\nabla \phi_3, \nabla \phi_4) &= \frac{1}{2}(1 - \phi_3^2 - \phi_4^2). \end{aligned} \quad (9)$$

and moreover

**Lemma 4.2.**

$$\begin{aligned} 3 \int \omega(\nabla \phi_1, \nabla \phi_2) &= 3 \int \phi^2 = \|\nabla \phi_1\|_{L^2} \|\nabla \phi_2\|_{L^2} = \|\nabla \phi\|_{L^2}^2 \\ 3 \int \omega(\nabla \phi_3, \nabla \phi_4) &= 3 \int \phi^2 = \|\nabla \phi_3\|_{L^2} \|\nabla \phi_4\|_{L^2} = \|\nabla \phi\|_{L^2}^2 \\ -3 \int \omega(\nabla \phi_i, \nabla \phi_j) &= 0 \quad \text{for other } ij \\ -3 \int \phi_k \omega(\nabla \phi_i, \nabla \phi_j) &= 0 \quad \forall i, j, k \\ -3 \int \phi_1^2 \omega(\nabla \phi_1, \nabla \phi_2) &= -3 \int \phi_2^2 \omega(\nabla \phi_1, \nabla \phi_2) = -\frac{1}{2} \int \phi^2 \\ -3 \int \phi_3^2 \omega(\nabla \phi_1, \nabla \phi_2) &= -3 \int \phi_4^2 \omega(\nabla \phi_1, \nabla \phi_2) = -\int \phi^2 \\ -3 \int \phi_1^2 \omega(\nabla \phi_3, \nabla \phi_4) &= -3 \int \phi_2^2 \omega(\nabla \phi_3, \nabla \phi_4) = -\int \phi^2 \\ -3 \int \phi_3^2 \omega(\nabla \phi_3, \nabla \phi_4) &= -3 \int \phi_4^2 \omega(\nabla \phi_3, \nabla \phi_4) = -\frac{1}{2} \int \phi^2 \\ -3 \int \phi_1 \phi_4 \omega(\nabla \phi_1, \nabla \phi_3) &= -3 \int \phi_1 \phi_3 \omega(\nabla \phi_2, \nabla \phi_3) = -\frac{1}{4} \int \phi^2 \\ -3 \int \phi_1 \phi_3 \omega(\nabla \phi_1, \nabla \phi_4) &= -3 \int \phi_2 \phi_3 \omega(\nabla \phi_2, \nabla \phi_4) = \frac{1}{4} \int \phi^2 \\ -3 \int \phi_2 \phi_3 \omega(\nabla \phi_1, \nabla \phi_3) &= -3 \int \phi_2 \phi_4 \omega(\nabla \phi_1, \nabla \phi_4) = \frac{1}{4} \int \phi^2 \\ -3 \int \phi_2 \phi_4 \omega(\nabla \phi_2, \nabla \phi_3) &= -3 \int \phi_1 \phi_4 \omega(\nabla \phi_2, \nabla \phi_4) = -\frac{1}{4} \int \phi^2 \\ -3 \int \phi_i \phi_j \omega(\nabla \phi_k, \nabla \phi_s) &= 0 \quad \text{for other cases.} \end{aligned}$$

**Lemma 4.3.** *If  $f, h \in E_{\lambda_1}$ , that is  $f = \sum_i \mu_i \phi_i$ ,  $h = \sum_j \sigma_j \phi_j$ , for some constant  $\mu_i, \sigma_j$ , then (8) holds, with equality if and only if  $\sigma_2 = -\mu_1$ ,  $\sigma_1 = \mu_2$ ,  $\sigma_4 = -\mu_3$ ,  $\sigma_3 = \mu_4$ .*

**Proof.** Using the previous lemma,

$$\begin{aligned} -3 \int \omega(\nabla f, \nabla h) dM &= (\mu_1 \sigma_2 - \mu_2 \sigma_1) \int -3 \omega(\nabla \phi_1, \nabla \phi_2) + (\mu_3 \sigma_4 - \mu_4 \sigma_3) \int -3 \omega(\nabla \phi_3, \nabla \phi_4) \\ &= -(\mu_1 \sigma_2 - \mu_2 \sigma_1 + \mu_3 \sigma_4 - \mu_4 \sigma_3) \|\nabla \phi\|_{L^2}^2 \\ &\leq \frac{1}{2}(\sum_i \mu_i^2 + \sigma_i^2) \|\nabla \phi\|_{L^2}^2 = \frac{1}{2}(\|\nabla f\|_{L^2}^2 + \|\nabla h\|_{L^2}^2). \end{aligned}$$

The equality case follows immediately.  $\square$

**Lemma 4.4.** *If  $f, h \in E_{\lambda_2}$  are nonzero, then (8) holds with strict inequality.*

**Proof.** Set  $f = \sum_i \alpha_i \phi_i^2 + \sum_{i < j} A_{ij} \phi_i \phi_j$ , and  $h = \sum_i \beta_i \phi_i^2 + \sum_{i < j} B_{ij} \phi_i \phi_j$ , where  $\alpha_i, A_{ij}, \beta_i, B_{ij}$  are constants. Now we compute

$$\begin{aligned}
-3 \int \omega(\nabla f, \nabla h) = & \\
& -3 \int \omega(\nabla \phi_1, \nabla \phi_2) [(2\alpha_1 \phi_1 + A_{12} \phi_2 + A_{13} \phi_3 + A_{14} \phi_4)(2\beta_2 \phi_2 + B_{12} \phi_1 + B_{23} \phi_3 + B_{24} \phi_4) \\
& \quad - (2\alpha_2 \phi_2 + A_{12} \phi_1 + A_{23} \phi_3 + A_{24} \phi_4)(2\beta_1 \phi_1 + B_{12} \phi_2 + B_{13} \phi_3 + B_{14} \phi_4)] \\
& -3 \int \omega(\nabla \phi_1, \nabla \phi_3) [(2\alpha_1 \phi_1 + A_{12} \phi_2 + A_{13} \phi_3 + A_{14} \phi_4)(2\beta_3 \phi_3 + B_{13} \phi_1 + B_{23} \phi_2 + B_{34} \phi_4) \\
& \quad - (2\alpha_3 \phi_3 + A_{13} \phi_1 + A_{23} \phi_2 + A_{34} \phi_4)(2\beta_1 \phi_1 + B_{12} \phi_2 + B_{13} \phi_3 + B_{14} \phi_4)] \\
& -3 \int \omega(\nabla \phi_1, \nabla \phi_4) [(2\alpha_1 \phi_1 + A_{12} \phi_2 + A_{13} \phi_3 + A_{14} \phi_4)(2\beta_4 \phi_4 + B_{14} \phi_1 + B_{24} \phi_2 + B_{34} \phi_3) \\
& \quad - (2\alpha_4 \phi_4 + A_{14} \phi_1 + A_{24} \phi_2 + A_{34} \phi_3)(2\beta_1 \phi_1 + B_{12} \phi_2 + B_{13} \phi_3 + B_{14} \phi_4)] \\
& -3 \int \omega(\nabla \phi_2, \nabla \phi_3) [(2\alpha_2 \phi_2 + A_{12} \phi_1 + A_{23} \phi_3 + A_{24} \phi_4)(2\beta_3 \phi_3 + B_{13} \phi_1 + B_{23} \phi_2 + B_{34} \phi_4) \\
& \quad - (2\alpha_3 \phi_3 + A_{13} \phi_1 + A_{23} \phi_2 + A_{34} \phi_4)(2\beta_2 \phi_2 + B_{12} \phi_1 + B_{24} \phi_4 + B_{23} \phi_3)] \\
& -3 \int \omega(\nabla \phi_2, \nabla \phi_4) [(2\alpha_2 \phi_2 + A_{12} \phi_1 + A_{23} \phi_3 + A_{24} \phi_4)(2\beta_4 \phi_4 + B_{14} \phi_1 + B_{24} \phi_2 + B_{34} \phi_3) \\
& \quad - (2\alpha_4 \phi_4 + A_{14} \phi_1 + A_{24} \phi_2 + A_{34} \phi_3)(2\beta_2 \phi_2 + B_{12} \phi_1 + B_{24} \phi_4 + B_{23} \phi_3)] \\
& -3 \int \omega(\nabla \phi_3, \nabla \phi_4) [(2\alpha_3 \phi_3 + A_{13} \phi_1 + A_{23} \phi_2 + A_{34} \phi_4)(2\beta_4 \phi_4 + B_{14} \phi_1 + B_{24} \phi_2 + B_{34} \phi_3) \\
& \quad - (2\alpha_4 \phi_4 + A_{14} \phi_1 + A_{24} \phi_2 + A_{34} \phi_3)(2\beta_3 \phi_3 + B_{13} \phi_1 + B_{23} \phi_2 + B_{34} \phi_4)].
\end{aligned}$$

Thus, using Lemma 4.2,

$$\begin{aligned}
-3 \int \omega(\nabla f, \nabla h) = & \\
& -3 \int \omega(\nabla \phi_1, \nabla \phi_2) [2\alpha_1 B_{12} \phi_1^2 + 2\beta_2 A_{12} \phi_2^2 + A_{13} B_{23} \phi_3^2 + A_{14} B_{24} \phi_4^2 \\
& \quad - 2\beta_1 A_{12} \phi_1^2 - 2\alpha_2 B_{12} \phi_2^2 - A_{23} B_{13} \phi_3^2 - A_{24} B_{14} \phi_4^2] \\
& -3 \int \omega(\nabla \phi_3, \nabla \phi_4) [A_{13} B_{14} \phi_1^2 + A_{23} B_{24} \phi_2^2 + 2\alpha_3 B_{34} \phi_3^2 + 2\beta_4 A_{34} \phi_4^2 \\
& \quad - A_{14} B_{13} \phi_1^2 - A_{24} B_{23} \phi_2^2 - 2\beta_3 A_{34} \phi_3^2 - 2\alpha_4 B_{34} \phi_4^2] \\
& -3 \int \omega(\nabla \phi_1, \nabla \phi_3) [2\alpha_1 B_{34} \phi_1 \phi_4 + A_{14} B_{13} \phi_1 \phi_4 - A_{13} B_{14} \phi_1 \phi_4 - 2\beta_1 A_{34} \phi_1 \phi_4 \\
& \quad + 2\beta_3 A_{12} \phi_2 \phi_3 + A_{13} B_{23} \phi_2 \phi_3 - A_{23} B_{13} \phi_2 \phi_3 - 2\alpha_3 B_{12} \phi_2 \phi_3] \\
& -3 \int \omega(\nabla \phi_1, \nabla \phi_4) [2\alpha_1 B_{34} \phi_1 \phi_3 + A_{13} B_{14} \phi_1 \phi_3 - A_{14} B_{13} \phi_1 \phi_3 - 2\beta_1 A_{34} \phi_1 \phi_3 \\
& \quad + 2\beta_4 A_{12} \phi_2 \phi_4 + A_{14} B_{24} \phi_2 \phi_4 - A_{24} B_{14} \phi_2 \phi_4 - 2\alpha_4 B_{12} \phi_2 \phi_4] \\
& -3 \int \omega(\nabla \phi_2, \nabla \phi_3) [2\beta_3 A_{12} \phi_1 \phi_3 + A_{23} B_{13} \phi_1 \phi_3 - A_{13} B_{23} \phi_1 \phi_3 - 2\alpha_3 B_{12} \phi_1 \phi_3 \\
& \quad + 2\alpha_2 B_{34} \phi_2 \phi_4 + A_{24} B_{23} \phi_2 \phi_4 - A_{23} B_{24} \phi_2 \phi_4 - 2\beta_2 A_{34} \phi_2 \phi_4] \\
& -3 \int \omega(\nabla \phi_2, \nabla \phi_4) [2\beta_4 A_{12} \phi_1 \phi_4 + A_{24} B_{14} \phi_1 \phi_4 - A_{14} B_{24} \phi_1 \phi_4 - 2\alpha_4 B_{12} \phi_1 \phi_4 \\
& \quad + 2\alpha_2 B_{34} \phi_2 \phi_3 + A_{23} B_{24} \phi_2 \phi_3 - A_{24} B_{23} \phi_2 \phi_3 - 2\beta_2 A_{34} \phi_2 \phi_3]
\end{aligned}$$

$$\begin{aligned}
&= \int \phi^2 \left\{ \begin{aligned} &-\frac{1}{2}[2\alpha_1 B_{12} + 2\beta_2 A_{12} - 2\beta_1 A_{12} - 2\alpha_2 B_{12} + 2\alpha_3 B_{34} + 2\beta_4 A_{34} - 2\beta_3 A_{34} - 2\alpha_4 B_{34}] \\ &-[A_{13}B_{23} + A_{14}B_{24} - A_{23}B_{13} - A_{24}B_{14} + A_{13}B_{14} + A_{23}B_{24} - A_{14}B_{13} - A_{24}B_{23}] \\ &+\frac{1}{4}[-2\alpha_1 B_{34} - A_{14}B_{13} + A_{13}B_{14} + 2\beta_1 A_{34} + 2\beta_3 A_{12} + A_{13}B_{23} - A_{23}B_{13} - 2\alpha_3 B_{12} \\ &\quad + 2\alpha_1 B_{34} + A_{13}B_{14} - A_{14}B_{13} - 2\beta_1 A_{34} + 2\beta_4 A_{12} + A_{14}B_{24} - A_{24}B_{14} - 2\alpha_4 B_{12} \\ &\quad - 2\beta_3 A_{12} - A_{23}B_{13} + A_{13}B_{23} + 2\alpha_3 B_{12} - 2\alpha_2 B_{34} - A_{24}B_{23} + A_{23}B_{24} + 2\beta_2 A_{34} \\ &\quad - 2\beta_4 A_{12} - A_{24}B_{14} + A_{14}B_{24} + 2\alpha_4 B_{12} + 2\alpha_2 B_{34} + A_{23}B_{24} - A_{24}B_{23} - 2\beta_2 A_{34}] \end{aligned} \right\} \\
&= \int \phi^2 \left\{ \begin{aligned} &-[\alpha_1 B_{12} + \beta_2 A_{12} - \beta_1 A_{12} - \alpha_2 B_{12} + \alpha_3 B_{34} + \beta_4 A_{34} - \beta_3 A_{34} - \alpha_4 B_{34}] \\ &-[A_{13}B_{23} + A_{14}B_{24} - A_{23}B_{13} - A_{24}B_{14} + A_{13}B_{14} + A_{23}B_{24} - A_{14}B_{13} - A_{24}B_{23}] \\ &+\frac{1}{2}[-A_{14}B_{13} + A_{13}B_{14} + A_{13}B_{23} - A_{23}B_{13} + A_{14}B_{24} - A_{24}B_{14} - A_{24}B_{23} + A_{23}B_{24}] \end{aligned} \right\} \\
&= \int \phi^2 \left\{ \begin{aligned} &[-\alpha_1 B_{12} - \beta_2 A_{12} + \beta_1 A_{12} + \alpha_2 B_{12} - \alpha_3 B_{34} - \beta_4 A_{34} + \beta_3 A_{34} + \alpha_4 B_{34}] \\ &+\frac{1}{2}[-A_{13}B_{23} - A_{14}B_{24} + A_{23}B_{13} + A_{24}B_{14} - A_{13}B_{14} - A_{23}B_{24} + A_{14}B_{13} + A_{24}B_{23}] \end{aligned} \right\}
\end{aligned}$$

and applying the same lemmas we see that

$$\|\nabla f\|_{L^2}^2 = \left[ 2\left(\sum_k \alpha_k^2\right) - \frac{4}{3}\left(\sum_{i<j} \alpha_i \alpha_j\right) + \frac{4}{3}\left(\sum_{i<j} A_{ij}^2\right) \right] \int \phi^2.$$

Hence, we have to verify if the following inequality is true:

$$[-\alpha_1 B_{12} - \beta_2 A_{12} + \beta_1 A_{12} + \alpha_2 B_{12} - \alpha_3 B_{34} - \beta_4 A_{34} + \beta_3 A_{34} + \alpha_4 B_{34}] \quad (10)$$

$$+\frac{1}{2}[-A_{13}B_{23} - A_{14}B_{24} + A_{23}B_{13} + A_{24}B_{14} - A_{13}B_{14} - A_{23}B_{24} + A_{14}B_{13} + A_{24}B_{23}] \quad (11)$$

$$+\frac{2}{3}\left(\sum_{i<j} \alpha_i \alpha_j + \beta_i \beta_j\right) \quad (12)$$

$$\leq \sum_k (\alpha_k^2 + \beta_k^2) + \frac{2}{3}\left(\sum_{i<j} A_{ij}^2 + B_{ij}^2\right). \quad (13)$$

This is equivalent to prove the inequalities

$$(11) \leq \frac{2}{3}(A_{13}^2 + A_{14}^2 + A_{23}^2 + A_{24}^2 + B_{13}^2 + B_{14}^2 + B_{23}^2 + B_{24}^2) \quad (14)$$

$$(10) + (12) \leq \sum_k (\alpha_k^2 + \beta_k^2) + \frac{2}{3}(A_{12}^2 + A_{34}^2 + B_{12}^2 + B_{34}^2). \quad (15)$$

Note that

$$\begin{aligned}
2 \times (11) &\leq (A_{13}^2 + A_{14}^2 + A_{23}^2 + A_{24}^2 + B_{13}^2 + B_{14}^2 + B_{23}^2 + B_{24}^2) \\
&\leq \frac{4}{3}(A_{13}^2 + A_{14}^2 + A_{23}^2 + A_{24}^2 + B_{13}^2 + B_{14}^2 + B_{23}^2 + B_{24}^2),
\end{aligned}$$

and so inequality (14) holds, with equality if and only if

$$A_{13} = A_{14} = A_{23} = A_{24} = B_{13} = B_{14} = B_{23} = B_{24} = 0.$$

Now

$$\begin{aligned}
3 \times (10) &= 3(\alpha_2 - \alpha_1)B_{12} - 3(\beta_2 - \beta_1)A_{12} + 3(\alpha_4 - \alpha_3)B_{34} + 3(-\beta_4 + \beta_3)A_{34} \\
&\leq \frac{3}{2}((\alpha_2 - \alpha_1)^2 + (\beta_2 - \beta_1)^2 + (\alpha_4 - \alpha_3)^2 + (-\beta_4 + \beta_3)^2) \\
&\quad + \frac{3}{2}(A_{12}^2 + A_{34}^2 + B_{12}^2 + B_{34}^2) \\
&\leq \frac{3}{2}((\alpha_2 - \alpha_1)^2 + (\beta_2 - \beta_1)^2 + (\alpha_4 - \alpha_3)^2 + (-\beta_4 + \beta_3)^2) \tag{16}
\end{aligned}$$

$$+ 2(A_{12}^2 + A_{34}^2 + B_{12}^2 + B_{34}^2). \tag{17}$$

We will prove that

$$(16) + 3 \times (12) \leq \sum_k 3(\alpha_k^2 + \beta_k^2), \tag{18}$$

with equality iff  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$  and  $\beta_1 = \beta_2 = \beta_3 = \beta_4$ , which proves that (15) holds. Furthermore, from (17) we see that equality in (15) is achieved iff

$$A_{12} = A_{34} = B_{12} = B_{34} = 0, \quad \text{and for all } i, j \quad \alpha_i = \alpha_j, \quad \beta_i = \beta_j.$$

In order to prove (18) we only have to show that

$$\frac{3}{2}((\alpha_2 - \alpha_1)^2 + (\alpha_4 - \alpha_3)^2) + 2 \sum_{i < j} \alpha_i \alpha_j \leq 3 \sum_k \alpha_k^2,$$

or equivalently, that

$$-2\alpha_1\alpha_2 - 2\alpha_3\alpha_4 + 4\alpha_1\alpha_3 + 4\alpha_1\alpha_4 + 4\alpha_2\alpha_3 + 4\alpha_2\alpha_4 \leq 3 \sum_k \alpha_k^2.$$

But this is just

$$(\alpha_1 - \alpha_3)^2 + (\alpha_3 - \alpha_2)^2 + (\alpha_2 - \alpha_4)^2 + (\alpha_4 - \alpha_1)^2 + (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)^2 \geq 0,$$

with equality to zero iff  $\alpha_i = \alpha_j \forall i, j$ . We have proved that inequality (8) is satisfied, with equality iff  $f = \alpha(\sum_k \phi_k^2) = \alpha$  constant and  $h$  constant, and so they must vanish.  $\square$

Theorem 1.1, with Lemmas 4.1, 4.3 and 4.4, prove that (8) holds for any pair of functions  $(f, h)$ , and so Theorem 1.2 is proved. Corollary 1.1 follows from these lemmas.

In (Salavessa, 2010, Theorem 4.2) a uniqueness theorem was obtained, on a class of closed  $m$ -dimensional submanifolds with parallel mean curvature and calibrated extended tangent in a Euclidean space  $\mathbb{R}^{m+n}$ , and satisfying an integral height inequality. We will recall such results for the case  $\Omega$  parallel. We denote by  $B^\nu$  the  $\nu$ -component of the second fundamental form  $B$  and by  $B^F$  the  $F$ -component,  $B = B^\nu + B^F$ , where  $F$  is the orthogonal complement of  $\nu$  in the normal bundle.

**Theorem 4.1.** *If  $\Omega$  is a parallel calibration of rank  $(m + 1)$  on  $\mathbb{R}^{m+n}$ , and  $\phi : M \rightarrow \mathbb{R}^{m+n}$  is an immersed closed  $\Omega$ -stable  $m$ -dimensional submanifold with parallel mean curvature and calibrated extended tangent space, and*

$$\int_M S(2 + h\|H\|)dM \leq 0, \quad (19)$$

*where  $h = \langle \phi, \nu \rangle$  and  $S = \sum_{ij} \langle \phi, (B(e_i, e_j))^F \rangle B^\nu(e_i, e_j)$ , then  $\phi$  is pseudo-umbilical and  $S = 0$ . Furthermore, if  $NM$  is a trivial bundle, then the minimal calibrated extension of  $M$  is a Euclidean space  $\mathbb{R}^{m+1}$ , and  $M$  is a Euclidean  $m$ -sphere.*

Theorem 1.3 is an immediate consequence of Theorem 1.2 and the above theorem.

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